

SUPERPOSITION IN THE P-LAPLACE EQUATION.

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ABSTRACT. That a superposition of fundamental solutions to the p -Laplace Equation is p -superharmonic – even in the non-linear cases $p > 2$ – has been known since M. Crandall and J. Zhang published their paper *Another Way to Say Harmonic* in 2003. We give a simple proof and extend the result by means of an explicit formula for the p -Laplacian of the superposition.

1. INTRODUCTION

Our object is a superposition of fundamental solutions for the p -Laplace Equation

$$(1.1) \quad \Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u) = 0.$$

Although the equation is non-linear, the function

$$V(x) = \int_{\mathbb{R}^n} \frac{\rho(y)}{|x - y|^{\frac{n-p}{p-1}}} dy, \quad \rho \geq 0, \quad 2 \leq p < n$$

is a supersolution in \mathbb{R}^n , i.e. $\Delta_p V \leq 0$ in the sense of distributions. It is a so-called p -superharmonic function – see Definition 2 on page 6 – according to which it has to obey the comparison principle. The case $p = 2$ reduces to the Laplace Equation $\Delta u = 0$ with the Newtonian potential

$$V(x) = \int_{\mathbb{R}^n} \frac{\rho(y)}{|x - y|^{n-2}} dy,$$

which is a superharmonic function.

M. Crandall and J. Zhang discovered in [CZ03] that the sum

$$\sum_{i=1}^N \frac{a_i}{|x - y_i|^{\frac{n-p}{p-1}}}, \quad a_i > 0$$

of fundamental solutions is a p -superharmonic function. Their proof was written in terms of viscosity supersolutions. A different proof was given in [LM08]. The purpose of our note is a *simple* proof of the following theorem:

Theorem 1. *Let $2 \leq p < n$. For an arbitrary concave function K ,*

$$(1.2) \quad W(x) := \sum_{i=1}^{\infty} \frac{a_i}{|x - y_i|^{\frac{n-p}{p-1}}} + K(x), \quad y_i \in \mathbb{R}^n, a_i \geq 0,$$

is p -superharmonic in \mathbb{R}^n , provided the series converges at some point.

Through Riemann sums one can also include potentials like

$$\int_{\mathbb{R}^n} \frac{\rho(y)}{|x - y|^{\frac{n-p}{p-1}}} dy + K(x), \quad \rho \geq 0.$$

Similar results are given for the cases $p = n$ and $p > n$ and, so far as we know, the extra concave term $K(x)$ is a new feature. The key aspect of the proof is the explicit formula (3.2) for the p -Laplacian of the superposition. Although the formula is easily obtained, it seems to have escaped attention up until now.

Finally, we mention that in [GT10] the superposition of fundamental solutions has been extended to the p -Laplace Equation in the Heisenberg group. (Here one of the variables is discriminated.) In passing, we show in Section 6 that similar results are *not* valid for the evolutionary equations

$$\frac{\partial}{\partial t} u = \Delta_p u \quad \text{and} \quad \frac{\partial}{\partial t} (|u|^{p-2} u) = \Delta_p u$$

where $u = u(x, t)$. We are able to bypass a lengthy calculation in our counter examples.

2. THE FUNDAMENTAL SOLUTION

Consider a radial function, say

$$f(x) = v(|x|)$$

where we assume that $v \in C^2(0, \infty)$. By differentiation

$$(2.1) \quad \begin{aligned} \nabla f &= \frac{v'}{|x|} x^T, & |\nabla f| &= |v'|, \\ \mathcal{H}f &= v'' \frac{xx^T}{|x|^2} + \frac{v'}{|x|} \left(I - \frac{xx^T}{|x|^2} \right), & \Delta f &= v'' + (n-1) \frac{v'}{|x|}, \end{aligned}$$

when $x \neq 0$.

The Rayleigh quotient formed by the Hessian matrix $\mathcal{H}f = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]$ above will play a central role. Notice that for any non-zero $z \in \mathbb{R}^n$, we have that

$$\frac{z^T}{|z|} \frac{xx^T}{|x|^2} \frac{z}{|z|} = \cos^2 \theta$$

where θ is the angle between the two vectors x and z . This yields the expedient formula

$$(2.2) \quad \frac{z^T(\mathcal{H}f)z}{|z|^2} = v'' \cos^2 \theta + \frac{v'}{|x|} \sin^2 \theta, \quad x, z \neq 0.$$

Since the gradient of a radial function is parallel to x , the Rayleigh quotient in the identity

$$(2.3) \quad \operatorname{div} (|\nabla f|^{p-2} \nabla f) = |\nabla f|^{p-2} \left((p-2) \frac{\nabla f(\mathcal{H}f) \nabla f^T}{|\nabla f|^2} + \Delta f \right)$$

reduces to v'' . The vanishing of the whole expression is then equivalent to

$$(2.4) \quad (p-1)v'' + (n-1) \frac{v'}{|x|} = 0$$

which, integrated once, implies that a radially decreasing solution w is on the form

$$(2.5) \quad w(x) = v(|x|) \quad \text{where} \quad v'(|x|) = -c|x|^{\frac{1-n}{p-1}}.$$

The constant $c = c_{n,p} > 0$ can now be chosen so that

$$\Delta_p w + \delta = 0$$

in the sense of distributions. Thus

$$(2.6) \quad w(x) = \begin{cases} -c_{n,p} \frac{p-1}{p-n} |x|^{\frac{p-n}{p-1}}, & \text{when } p \neq n, \\ -c_{n,n} \ln |x|, & \text{when } p = n \end{cases}$$

is the **fundamental solution** to the p -Laplace Equation (1.1).

3. SUPERPOSITION OF FUNDAMENTAL SOLUTIONS

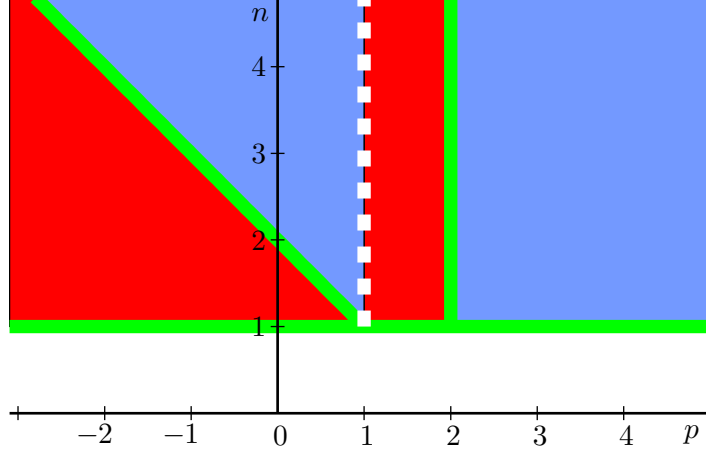
We now form a superposition of translates of the fundamental solution and compute its p -Laplacian. To avoid convergence issues all sums are, for the moment, assumed finite.

Lemma 1. *Let w be the fundamental solution to the p -Laplace equation. Define the function V as*

$$(3.1) \quad V(x) := \sum_{i=1}^N a_i w(x - y_i), \quad a_i > 0, y_i \in \mathbb{R}^n.$$

Then, in any dimension and for any $p \neq 1$ ¹, $\Delta_p V$ is of the same sign wherever it is defined in \mathbb{R}^n . Furthermore, the dependence of the sign on p and n is as indicated in figure 1.

¹When $p = 1$ there are no non-constant radial solutions of (1.1). Instead we get the zero mean curvature equation in which a solution's level sets are minimal surfaces.

FIGURE 1. $\Delta_p V \leq 0$, $\Delta_p V = 0$, $\Delta_p V \geq 0$

Proof. We simplify the notation by letting w_i and v_i denote that the functions w and v are to be evaluated at $x - y_i$ and $|x - y_i|$, respectively.

First, the linearity of the Hessian and the Laplacian enable us to write

$$\begin{aligned} \Delta_p V &= |\nabla V|^{p-2} \left((p-2) \frac{\nabla V (\mathcal{H}V) \nabla^T V}{|\nabla V|^2} + \Delta V \right) \\ &= |\nabla V|^{p-2} \sum_{i=1}^N a_i \left((p-2) \frac{\nabla V (\mathcal{H}w_i) \nabla^T V}{|\nabla V|^2} + \Delta w_i \right). \end{aligned}$$

Secondly, by (2.1) and (2.2) this is

$$\begin{aligned} &= |\nabla V|^{p-2} \sum_{i=1}^N a_i \left((p-2) \left(v_i'' \cos^2 \theta_i + \frac{v_i'}{|x - y_i|} \sin^2 \theta_i \right) \right. \\ &\quad \left. + v_i'' + (n-1) \frac{v_i'}{|x - y_i|} \right) \\ &= |\nabla V|^{p-2} \sum_{i=1}^N a_i \left((p-2) \left(\frac{v_i'}{|x - y_i|} - v_i'' \right) \sin^2 \theta_i \right. \\ &\quad \left. + (p-1)v_i'' + (n-1) \frac{v_i'}{|x - y_i|} \right) \end{aligned}$$

where θ_i is the angle between $x - y_i$ and $\nabla V(x)$. And finally, as w is a fundamental solution, the last two terms disappear by (2.4). We get

$$\Delta_p V = (p-2) |\nabla V|^{p-2} \sum_{i=1}^N a_i \left(\frac{v_i'}{|x - y_i|} - v_i'' \right) \sin^2 \theta_i.$$

It only remains to use the formula (2.5) for v'_i to compute that

$$\frac{v'_i}{|x - y_i|} - v''_i = -c_{n,p} \frac{p + n - 2}{p - 1} |x - y_i|^{\frac{2-n-p}{p-1}}$$

and the sign of $\Delta_p V$ can easily be read off the final identity

$$(3.2) \quad \Delta_p V(x) = -c_{n,p} \frac{(p-2)(p+n-2)}{p-1} |\nabla V|^{p-2} \sum_{i=1}^N a_i \frac{\sin^2 \theta_i}{|x - y_i|^{\frac{p+n-2}{p-1}}}.$$

□

Remark 1. The three green lines in figure 1 deserve some attention. The line $p = 2$ is obvious since the equation becomes linear. So is the line $n = 1$ as the “angle” between two numbers is 0 or π . The little surprise, perhaps, is the case $p + n = 2$. Then the terms in V will be on the form $a_i |x - y_i|^2$ and it all reduces to the rather unexciting explanation that a linear combination of quadratics is again a quadratic.

4. ADDING MORE TERMS

We will now examine what will happen to the sign of the p -Laplace operator when an extra term, $K(x)$, is added to the linear combination (3.1). We will from now on only consider $p > 2$. Restricted to this case, the factor $C_{n,p} := c_{n,p} \frac{(p-2)(p+n-2)}{p-1}$ in (3.2) stays positive.

Let V be as in Lemma 1 and let $K \in C^2$. For efficient notation, write $\xi = \xi(x) := \nabla V(x) + \nabla K(x)$. Then

$$\begin{aligned} \Delta_p(V + K) &= |\xi|^{p-2} \left((p-2) \frac{\xi \mathcal{H}(V + K) \xi^T}{|\xi|^2} + \Delta(V + K) \right) \\ &= |\xi|^{p-2} \left((p-2) \frac{\xi (\mathcal{H}V) \xi^T}{|\xi|^2} + \Delta V \right) \\ &\quad + |\xi|^{p-2} \left((p-2) \frac{\xi (\mathcal{H}K) \xi^T}{|\xi|^2} + \Delta K \right). \end{aligned}$$

Now, the second to last term equals

$$-C_{n,p} |\xi|^{p-2} \sum_i a_i |x - y_i|^{\frac{2-n-p}{p-1}} \sin^2 \alpha_i \leq 0$$

where α_i is the angle between $x - y_i$ and $\nabla V(x) + \nabla K(x)$. Thus it suffices to ensure that the last term also is non-positive in order for the p -Laplace to hold its sign. Lemma 2 presents a sufficient condition.

Lemma 2. *Let $p > 2$ and define V as in (3.1). Then*

$$(4.1) \quad \Delta_p(V(x) + K(x)) \leq 0$$

for all concave functions $K \in C^2(\mathbb{R}^n)$ wherever the left-hand side is defined.

Proof. $z^T(\mathcal{H}K)z \leq 0$ for all $z \in \mathbb{R}^n$ since the Hessian matrix of a concave function K is negative semi-definite. Also K is superharmonic since the eigenvalues of $\mathcal{H}K$ are all non-positive, i.e. $\Delta K \leq 0$. Therefore,

$$\Delta_p(V(x) + K(x)) \leq |\xi|^{p-2} \left((p-2) \frac{\xi(\mathcal{H}K)\xi^T}{|\xi|^2} + \Delta K \right) \leq 0.$$

□

Remark 2. Though $K \in C^2$ being concave is sufficient, it is not necessary. A counter example is provided by the quadratic form

$$K(x) = \frac{1}{2}x^T A x, \quad \text{where } A = \text{diag}(1-m, 1, \dots, 1), \quad m = p+n-2.$$

Then K is not concave, but a calculation will confirm that $(p-2) \frac{\xi(\mathcal{H}K)\xi^T}{|\xi|^2} + \Delta K \leq 0$ and hence $\Delta_p(V+K) \leq 0$. In fact, a stronger result than Lemma 2 is possible: Let f_i be C^2 at x for $i = 1, \dots, N$ and let

$$\lambda_1^i \leq \lambda_2^i \leq \dots \leq \lambda_n^i$$

be the eigenvalues of the Hessian matrix $\mathcal{H}f_i(x)$. If

$$\lambda_1^i + \dots + \lambda_{n-1}^i + (p-1)\lambda_n^i \leq 0 \quad \forall i,$$

then $\Delta_p(\sum_i f_i) \leq 0$ at x .

5. p -SUPERHARMONICITY

We now prove that

$$W(x) := \sum_{i=1}^{\infty} a_i w(x - y_i) + K(x), \quad a_i \geq 0, y_i \in \mathbb{R}^n, \quad K \text{ concave}$$

is a p -superharmonic function in \mathbb{R}^n . The three cases $2 < p < n$, $p = n$ and $p > n$ are different and an additional assumption, (5.3), seems to be needed when $p \geq n$. In the first case, only convergence at one point is assumed. We start with the relevant definitions and a useful Dini-type lemma.

Definition 1. Let Ω be a domain in \mathbb{R}^n . A continuous function $h \in W_{loc}^{1,p}(\Omega)$ is **p -harmonic** if

$$(5.1) \quad \int |\nabla h|^{p-2} \nabla h \nabla \phi^T dx = 0$$

for each $\phi \in C_0^\infty(\Omega)$.

Definition 2. A function $u: \Omega \rightarrow (-\infty, \infty]$ is **p -superharmonic** in Ω if

- i) $u \not\equiv \infty$.
- ii) u is lower semi-continuous in Ω .
- iii) If $D \subset\subset \Omega$ and $h \in C(\overline{D})$ is p -harmonic in D with $h|_{\partial D} \leq u|_{\partial D}$, then $h \leq u$ in D .

Furthermore, if $u \in C^2(\Omega)$, it is a standard result that u is p -harmonic if and only if $\Delta_p u = 0$ and u is p -superharmonic if and only if $\Delta_p u \leq 0$.

Also, a function u in $C(\mathbb{R}^n) \cap W_{loc}^{1,p}(\mathbb{R}^n)$ is p -superharmonic if

$$(5.2) \quad \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \phi^T dx \geq 0$$

for all $0 \leq \phi \in C_0^\infty(\mathbb{R}^n)$. See [Lin86].

Lemma 3. *Let (f_N) be an increasing sequence of lower semi-continuous (l.s.c.) functions defined on a compact set C converging point-wise to a function $f \geq 0$. Then, given any $\epsilon > 0$ there is an $N_\epsilon \in \mathbb{N}$ such that*

$$f_N(x) > -\epsilon$$

for all $x \in C$ and all $N \geq N_\epsilon$.

The standard proof is omitted.

In the following, K is any concave function in \mathbb{R}^n . We let K_δ , $\delta > 0$ denote the smooth convolution $\phi_\delta * K$ with some mollifier ϕ_δ . One can show that K_δ is concave and

$$K_\delta \rightarrow K$$

locally uniformly on \mathbb{R}^n as $\delta \rightarrow 0^+$.

5.1. The case $2 < p < n$. Let $\delta > 0$. If $y_i \in \mathbb{R}^n$ and $a_i > 0$, the function

$$W_N^\delta(x) := \sum_{i=1}^N \frac{a_i}{|x - y_i|^{\frac{n-p}{p-1}}} + K_\delta(x)$$

is p -superharmonic except possibly at the poles y_i (Lemma 2). Defining $W_N^\delta(y_i) := \infty$, **we claim that W_N^δ is p -superharmonic in the whole \mathbb{R}^n .**

We have to verify Def. 2. Clearly, i) and ii) are valid. For the comparison principle in iii) we select $D \subset \subset \mathbb{R}^n$ (i.e. D is bounded) and let $h \in C(\overline{D})$ be p -harmonic in D with $h|_{\partial D} \leq W_N^\delta|_{\partial D}$. If any, isolate the points y_i in \overline{D} with ϵ -balls $B_i := B(y_i, \epsilon)$ where $\epsilon > 0$ is so small so that $W_N^\delta|_{B_i} \geq \max_{\overline{D}} h$. This is possible because h is bounded and because $\lim_{x \rightarrow y_i} W_N^\delta(x) = \infty$. Then W_N^δ is C^2 on $D \setminus \cup B_i$ so, by Lemma 2, $\Delta_p W_N^\delta \leq 0$ on this set. Also, $h|_{\partial(D \setminus \cup B_i)} \leq W_N^\delta|_{\partial(D \setminus \cup B_i)}$ by the construction of the ϵ -balls, so $h \leq W_N^\delta$ on this set since W_N^δ is p -superharmonic there. Naturally, $h \leq W_N^\delta$ on $\cup B_i$, so the inequality will hold in the whole domain D . This proves the claim.

Now $N \rightarrow \infty$. Assume that the limit function

$$W^\delta(x) := \sum_{i=1}^{\infty} \frac{a_i}{|x - y_i|^{\frac{n-p}{p-1}}} + K_\delta(x)$$

is finite at least at one point in \mathbb{R}^n . **We claim that W^δ is p -superharmonic.**

By assumption $W^\delta \not\equiv \infty$ and it is a standard result that the limit of an increasing sequence of l.s.c functions is l.s.c.

Part iii). Suppose that $D \subset\subset \mathbb{R}^n$ and $h \in C(\overline{D})$ is p -harmonic in D with $h|_{\partial D} \leq W^\delta|_{\partial D}$. Then $(W_N^\delta - h)$ is an increasing sequence of l.s.c. functions on the compact set ∂D with point-wise limit $(W^\delta - h)|_{\partial D} \geq 0$. If $\epsilon > 0$, then $(W_N^\delta - h)|_{\partial D} > -\epsilon$ for a sufficiently big N by Lemma 3. That is

$$(h - \epsilon)|_{\partial D} < W_N^\delta|_{\partial D}$$

so $(h - \epsilon)|_D \leq W_N^\delta|_D$ since $h - \epsilon$ is p -harmonic and W_N^δ is p -superharmonic. Finally, since $W_N^\delta \leq W^\delta$ we get

$$(h - \epsilon)|_D \leq W^\delta|_D$$

and as ϵ was arbitrary, the required inequality $h \leq W^\delta$ in D is obtained and the claim is proved.

Let $\delta \rightarrow 0$ and set

$$W(x) := \sum_{i=1}^{\infty} \frac{a_i}{|x - y_i|^{\frac{n-p}{p-1}}} + K(x).$$

We claim that W is p -superharmonic.

Part i) and ii) are immediate. For part iii), assume $D \subset\subset \mathbb{R}^n$ and $h \in C(\overline{D})$ is p -harmonic in D with $h|_{\partial D} \leq W|_{\partial D}$. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that

$$|K(x) - K_\delta(x)| < \epsilon$$

at every $x \in \overline{D}$. We have

$$W^\delta = W + K_\delta - K > W - \epsilon \geq h - \epsilon$$

on ∂D . And again, since $h - \epsilon$ is p -harmonic and W^δ is p -superharmonic, we get $W^\delta \geq h - \epsilon$ in D . Thus

$$W|_D \geq W^\delta|_D - \epsilon \geq h|_D - 2\epsilon.$$

This proves the claim, settles the case $2 < p < n$ and completes the proof of Theorem 1.

We now turn to the situation $p \geq n$ and introduce the assumption

$$(5.3) \quad A := \sum_{i=1}^{\infty} a_i < \infty.$$

5.2. **The case $p=n$.** Let $\delta > 0$. The partial sums

$$W_N^\delta(x) := - \sum_{i=1}^N a_i \ln |x - y_i| + K_\delta(x)$$

are p -superharmonic in \mathbb{R}^n by the same argument as in the case $2 < p < n$.

Let $N \rightarrow \infty$. **We claim that**

$$W^\delta(x) := - \sum_{i=1}^{\infty} a_i \ln |x - y_i| + K_\delta(x)$$

is p -superharmonic in \mathbb{R}^n provided the sum converges absolutely² at least at one point.

Assume for the moment that, given a radius $R > 0$, it is possible to find numbers C_i so that

$$(5.4) \quad \begin{aligned} & \ln |x - y_i| \leq C_i \text{ for all } x \in B_R := B(0, R), \text{ and} \\ & \text{the series } \sum_{i=1}^{\infty} a_i C_i =: S_R \text{ converges.} \end{aligned}$$

Define the sequence (f_N) in B_R by

$$f_N(x) := \sum_{i=1}^N (-a_i \ln |x - y_i| + a_i C_i) + K_\delta(x), \quad f(x) := \lim_{N \rightarrow \infty} f_N(x).$$

Then (f_N) is an increasing sequence of l.s.c functions implying that f is l.s.c. in B_R and that

$$W^\delta = f - S_R$$

is as well. Since R can be arbitrarily big, we conclude that W^δ does not take the value $-\infty$ and is l.s.c. in \mathbb{R}^n .

For part iii) we show that f obeys the comparison principle. Assume $D \subset\subset B_R$ and $h \in C(\overline{D})$ is p -harmonic in D with $h|_{\partial D} \leq f|_{\partial D}$. Then $(f_N - h)$ is an increasing sequence of l.s.c. functions on the compact set ∂D with point-wise limit

$$(f - h)|_{\partial D} \geq 0.$$

If $\epsilon > 0$, then $(f_N - h)|_{\partial D} > -\epsilon$ for a sufficiently big N by Lemma 3. That is

$$(h - \epsilon)|_{\partial D} < f_N|_{\partial D}$$

so $(h - \epsilon)|_D \leq f_N|_D$ since $h - \epsilon$ is p -harmonic and f_N is p -superharmonic. Finally, since $f_N \leq f$ we get

$$(h - \epsilon)|_D \leq f|_D$$

²Conditional convergence is not sufficient. A counter example is $a_i = 1/i^2$, $|y_i| = \exp((-1)^i i)$, yielding $W^\delta(x) = -\infty$ for all $y_i \neq x \neq 0$.

and as ϵ was arbitrary, the required inequality $h \leq f$ in D is obtained. Hence $W^\delta(x) = f(x) - S_R$ is a p -superharmonic function in any ball B_R .

The claim is now proved if we can establish the existence of the numbers C_i satisfying (5.4). By a change of variables we may assume that the convergence is at the origin. That is

$$L := \sum_{i=1}^{\infty} a_i |\ln |y_i|| < \infty.$$

We have

$$\begin{aligned} \ln |x - y_i| &\leq \ln(|x| + |y_i|) \\ &\leq \ln(2 \max\{|x|, |y_i|\}) \\ &= \max\{\ln |x|, \ln |y_i|\} + \ln 2, \end{aligned}$$

so

$$C_i := \max\{\ln R, \ln |y_i|\} + \ln 2$$

will do since (for $R > 1/2$) the sequence of partial sums $\sum_{i=1}^N a_i C_i$ is increasing and bounded by $A \ln 2R + L$.

The final limit $\delta \rightarrow 0$ causes no extra problems.

$$W(x) := - \sum_{i=1}^{\infty} a_i \ln |x - y_i| + K(x)$$

is p -superharmonic in \mathbb{R}^n .

This settles the case $p = n$.

5.3. The case $p > n$. Let $\delta > 0$. Consider again the partial sums

$$W_N^\delta(x) := - \sum_{i=1}^N a_i |x - y_i|^{\frac{p-n}{p-1}} + K_\delta(x).$$

As before W_N^δ is p -superharmonic in \mathbb{R}^n , but now a different approach is required for the proof. For ease of notation, write

$$u(x) := - \sum_{i=1}^N a_i |x - y_i|^\alpha + K(x), \quad 0 < \alpha := \frac{p-n}{p-1} < 1,$$

where $K \in C^\infty(\mathbb{R}^n)$ is concave. We will show that u satisfies the integral inequality (5.2).

Clearly, u is continuous and $\int_\Omega |u|^p dx < \infty$ on any bounded domain Ω . Also,

$$|\nabla(|x|^\alpha)|^p = \left| \alpha \frac{x^T}{|x|^{2-\alpha}} \right|^p \propto \frac{1}{|x|^{(1-\alpha)p}}$$

where one can show that

$$(1 - \alpha)p < n.$$

Thus $\int |\nabla u|^p dx < \infty$ locally so $u \in C(\mathbb{R}^n) \cap W_{loc}^{1,p}(\mathbb{R}^n)$.

Let $0 \leq \phi \in C_0^\infty(\mathbb{R}^n)$ and write

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \phi^T dx &= \left(\int_{\mathbb{R}^n \setminus \cup_j B_j} + \int_{\cup_j B_j} \right) |\nabla u|^{p-2} \nabla u \nabla \phi^T dx \\ &=: I_\epsilon + J_\epsilon \end{aligned}$$

where $B_j := B(y_j, \epsilon)$ and where $\epsilon > 0$ is so small so that the balls are disjoint. Obviously, $J_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ but

$$\begin{aligned} I_\epsilon &= \int_{\partial(\mathbb{R}^n \setminus \cup_j B_j)} \phi |\nabla u|^{p-2} \nabla u \nu d\sigma - \int_{\mathbb{R}^n \setminus \cup_j B_j} \phi \Delta_p u dx \\ &\geq \int_{\cup_j \partial B_j} \phi |\nabla u|^{p-2} \nabla u \nu d\sigma \end{aligned}$$

since $\Delta_p u \leq 0$ on $\mathbb{R}^n \setminus \cup_j B_j$ by Lemma 2. Here, ν is a sphere's *inward* pointing normal so, for $x \in \partial B_i$,

$$\begin{aligned} \nabla u(x) \nu &= \nabla u(x) \frac{y_i - x}{\epsilon} \\ &= \left(-\alpha \sum_{j=1}^N a_j \frac{(x - y_j)^T}{|x - y_j|^{2-\alpha}} + \nabla K(x) \right) \frac{y_i - x}{\epsilon} \\ &= \frac{\alpha a_i}{\epsilon^{1-\alpha}} + \alpha \sum_{j \neq i} a_j \frac{(x - y_j)^T}{|x - y_j|^{2-\alpha}} \frac{x - y_i}{\epsilon} + \nabla K(x) \frac{y_i - x}{\epsilon} \\ &> \frac{\alpha a_i}{\epsilon^{1-\alpha}} - \frac{\alpha}{(d_i/2)^{1-\alpha}} \sum_{j \neq i} a_j - C_K, \quad d_i := \min_{j \neq i} |y_j - y_i| \\ &> 0 \end{aligned}$$

for ϵ sufficiently small.

That is,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \phi^T dx \geq 0$$

for all non-negative test-functions. The partial sums are therefore p -superharmonic functions.

Let $N \rightarrow \infty$ and set

$$W^\delta(x) := - \sum_{i=1}^{\infty} a_i |x - y_i|^\alpha + K_\delta(x), \quad \alpha := \frac{p-n}{p-1}$$

remembering the assumption (5.3). This function is automatically *upper* semi-continuous but as the definition of p -superharmonicity requires *lower* semi-continuity, *continuity* has to be shown.

We claim that W^δ is p -superharmonic in \mathbb{R}^n provided the series converges at least at some point.

Again we may assume that the convergence is at the origin. That is $\sum_{i=1}^{\infty} a_i |y_i|^\alpha < \infty$. Since $0 < \alpha < 1$, we get

$$\begin{aligned} |x - y_i|^\alpha &\leq (|x| + |y_i|)^\alpha \\ &\leq |x|^\alpha + |y_i|^\alpha \end{aligned}$$

so since

$$\sum_{i=1}^{\infty} a_i |x - y_i|^\alpha \leq |x|^\alpha \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} a_i |y_i|^\alpha < \infty$$

we see that $W_N^\delta \rightarrow W^\delta$ locally uniformly in \mathbb{R}^n . We infer that W^δ is continuous in \mathbb{R}^n .

For part iii), assume $D \subset\subset \mathbb{R}^n$ and $h \in C(\overline{D})$ is p -harmonic in D with $h|_{\partial D} \leq W^\delta|_{\partial D}$. Since $W^\delta \leq W_N^\delta$ and W_N^δ is p -superharmonic we get $h|_D \leq W_N^\delta|_D$ for all N . So given any $\epsilon > 0$

$$h|_D \leq W^\delta|_D + \epsilon$$

by uniformity on the bounded set D . This proves the claim.

Next, let $\delta \rightarrow 0$. **Then**

$$W(x) := - \sum_{i=1}^{\infty} a_i |x - y_i|^{\frac{p-n}{p-1}} + K(x)$$

is p -superharmonic in \mathbb{R}^n by the same argument as when $2 < p < n$. This settles the case $p > n$.

6. EPILOGUE: EVOLUTIONARY SUPERPOSITION.

The superposition of fundamental solutions has been extended to p -Laplace equations in the Heisenberg group, see [GT10]. When it comes to further extensions, a natural question is whether such a superposition is valid for the evolutionary p -Laplace equation

$$(6.1) \quad u_t = \Delta_p u,$$

or for the homogeneous equation

$$(6.2) \quad \frac{\partial}{\partial t}(|u|^{p-2}u) = \Delta_p u.$$

The following shows it does not.

In both cases $p > 2$ and $u = u(x, t)$ where $x \in \mathbb{R}^n$ and $t > 0$. The fundamental solutions to these equations are given by

$$\mathcal{B}(x, t) := \frac{1}{t^{n\beta}} \left(C - \frac{p-2}{p} \beta^{\frac{1}{p-1}} \left(\frac{|x|}{t^\beta} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}, \quad \beta := \frac{1}{n(p-2) + p}$$

and

$$\mathcal{W}(x, t) := \frac{c}{t^{\frac{n}{p(p-1)}}} \exp \left(-\frac{p-1}{p} (1/p)^{\frac{1}{p-1}} \left(\frac{|x|}{t^{1/p}} \right)^{\frac{p}{p-1}} \right)$$

respectively, where the subscript $+$ in the so-called Barenblatt solution $\mathcal{B}(x, t)$ means $(\cdot)_+ = \max\{\cdot, 0\}$. The C and c are positive constants chosen so that the solutions satisfy certain conservation properties. For any fixed positive time the functions are C^2 away from the origin and, in the case of \mathcal{B} , away from the boundary of its support. We also notice that $\mathcal{W} > 0$ on $\mathbb{R}^n \times (0, \infty)$ while $\mathcal{B} \geq 0$ has compact support for any finite t .

In some ways these functions are similar to the heat kernel. In particular, one can show that for any fixed $0 \neq y \in \mathbb{R}^n$ *there is a time when the time derivatives $\mathcal{W}_t(y, t)$ and $\mathcal{B}_t(y, t)$ change sign*. In fact, a calculation will confirm that

$$\Delta_p(a\mathcal{B}) - (a\mathcal{B})_t = (a^{p-1} - a)\mathcal{B}_t, \quad 0 < a \neq 1$$

changes sign at y when

$$|y| = (Cpn)^{\frac{p-1}{p}} \beta^{\frac{p-2}{p}} t^\beta$$

showing that not even the simple superposition $\mathcal{B} + \mathcal{B}$ holds. This counter example arises due to \mathcal{B} not being multiplicative and will not work when applied to \mathcal{W} .

Although the p -Laplacian

$$\Delta_p u = |\nabla u|^{p-2} \left((p-2) \frac{\nabla u(\mathcal{H}u) \nabla u^T}{|\nabla u|^2} + \Delta u \right), \quad p > 2,$$

is not well defined at x_0 if $\nabla u(x_0) = 0$, it can be continuously extended to zero if u is C^2 at the critical point. We will thus write $\Delta_p u(x_0) = 0$ in those cases.

Fix a non-zero $y \in \mathbb{R}^n$ and define the linear combination V as

$$(6.3) \quad V(x, t) := \mathcal{W}(x + y, t) + \mathcal{W}(x - y, t).$$

Since $\mathcal{W}(x, t) =: f(|x|, t)$ is radial in x , the gradient can be written as

$$\nabla \mathcal{W}(x, t) = f_1(|x|, t) \frac{x^T}{|x|}$$

and

$$V(0, t) = \mathcal{W}(y, t) + \mathcal{W}(-y, t) = 2\mathcal{W}(y, t).$$

Thus V is C^2 at the origin and

$$\nabla V(0, t) = \left|_{x=0} f_1(|x+y|, t) \frac{(x+y)^T}{|x+y|} + f_1(|x-y|, t) \frac{(x-y)^T}{|x-y|} = 0\right.$$

for all $t > 0$. So, at $x = 0$ we get

$$\begin{aligned} \frac{\partial}{\partial t} (|V|^{p-2} V) - \Delta_p V &= (p-1) V^{p-2} V_t - 0 \\ &= 2(p-1) (2\mathcal{W}(y, t))^{p-2} \mathcal{W}_t(y, t) \end{aligned}$$

which has the aforementioned change of sign at some time t . Thus the sum of the two fundamental solutions $\mathcal{W}(x \pm y, t)$ cannot be a supersolution nor a subsolution.

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